MATH4210: Financial Mathematics Tutorial 8

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Option Pricing of Continuous Market Models

Question

We consider an Asian option with payoff

$$g(S.) = \int_0^T S_t dt$$

under Black Scholes setting. Define

$S_t = S_0 \exp\left(\left(r - \frac{\sigma^2}{2}\right)t + \sigma B_t^{Q}\right).$

$$V_t = \mathbb{E}^{\mathbb{Q}}[e^{-r(T-t)} g(S.)|\mathcal{F}_t].$$

$\mathcal{F}_t = \sigma(S_\theta)_{\theta \in [0,t]}$

$\forall u \le t,\ S_u \in \mathcal{F}_t$

Show that

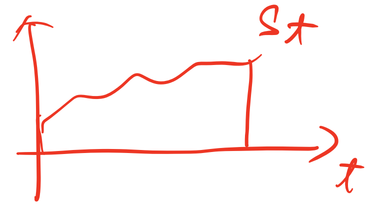
$$\begin{cases} V_t = \int_0^t S_s ds + (T-t)S_t & \text{if } r = 0 \\ V_t = e^{-r(T-t)} \int_0^t S_s ds + \frac{1}{r}(1 - e^{-r(T-t)})S_t & \text{if } r > 0 \end{cases}$$

For $t \geq 0$

$$V_t = E^{\mathbb{Q}}[\, e^{-r(T-t)} \, g(S_\cdot) \mid \mathcal{F}_t \,]$$

$$= E^{\mathbb{Q}}[\, e^{-r(T-t)} \cdot \int_0^T S_u \, du \mid \mathcal{F}_t \,]$$

$g: C([0,T], \mathbb{R}) \to \mathbb{R}$



Denote $(T-t) = \tau$.

$$V_t = e^{-r\tau} E[\int_0^T S_u \, du \mid \mathcal{F}_t]$$

$$= e^{-r\tau}\left(E[\int_0^t S_u \, du \mid \mathcal{F}_t] + E[\int_t^T S_u \, du \mid \mathcal{F}_t] \right)$$

$$= e^{-r\tau} \cdot \int_0^t E[S_u \mid \mathcal{F}_t] du + e^{-r\tau} \int_t^T E[S_u \mid \mathcal{F}_t] du$$

measurability

$$= e^{-r\tau} \int_0^t S_u \, du + e^{-r\tau} \int_t^T E[S_u \mid \mathcal{F}_t] du.$$

Since $S_t = S_0 \exp((r - \frac{\sigma^2}{2})t + \sigma B_t)$. $S_u = S_0 \exp(\cdots u + \sigma B_u)$.

$$E^{\mathbb{Q}}[S_u \mid \mathcal{F}_t] = E^{\mathbb{Q}}[S_u \mid S_t]$$

$$= E^{\mathbb{Q}}[S_t \cdot \exp((r - \frac{\sigma^2}{2}) \cdot (u-t) + \sigma(B_u^{\mathbb{Q}} - B_t^{\mathbb{Q}}) \mid S_t]$$

$$= S_t \, e^{(r - \frac{\sigma^2}{2})(u-t)} \cdot E^{\mathbb{Q}}[e^{\sigma(B_u^{\mathbb{Q}} - B_t^{\mathbb{Q}})}]$$

And $B_u^{\mathbb{Q}} - B_t^{\mathbb{Q}} \sim N(0, u-t) =: Z$

$$E^{\mathbb{Q}}[S_u \mid \mathcal{F}_t] = S_t \, e^{(r - \frac{\sigma^2}{2})(u-t)} \cdot E^{\mathbb{Q}}[e^{\sigma Z}]$$

$$= S_t \, e^{(r - \frac{\sigma^2}{2}) u - t} \cdot e^{\frac{1}{2} \cdot \sigma^2 \cdot (u-t)}$$

Recall the characteristic function or moment generating.

$$= S_t \, e^{r(u-t)}$$

$$\Rightarrow V_t = e^{-r\tau} \int_0^t S_u \, du + e^{-r\tau} \int_t^T S_t \cdot e^{r(u-t)} \cdot du$$

$$= e^{-r\tau}\int_0^t S_u du + \underbrace{S_t \cdot \int_t^T e^{r(u-T)} du}_{I}$$

① $r=0$, $I = S_t \cdot \int_t^T 1\, du$

$= S_t\,(T-t)$

② $r>0$ - - - - - -

$r=0$ *Alternative method*

Consider Π_t replicate V_t

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$$d\tilde{\Pi}_t = \phi_t\, d\tilde{S}_t \Rightarrow \tilde{\Pi}_t = \tilde{\Pi}_0 + \int_0^t \phi_u\, d\tilde{S}_u$$

$$\tilde{\Pi}_0 = \tilde{V}_0 = \phi_0 S_0$$

$$\Pi_T = V_T = \mathbb{E}\left[\int_0^T S_u du \,\middle|\, \mathcal{F}_T\right]$$

$$= \int_0^T S_u du$$

$$\begin{cases} \Pi_T = \Pi_0 + \int_0^T \phi_u\, dS_u \\ V_T = \int_0^T S_u du \\ \Pi_T = V_T \end{cases}$$

Consider $(\phi_t S_t)_{t\geq 0}$

$d(\phi_t S_t) = d\phi_t \cdot S_t + \phi_t\, dS_t$

$\Rightarrow \phi_t S_t - \phi_0 S_0 = \int_0^t S_u d\phi_u + \int_0^t \phi_u\, dS_u$

$$\Rightarrow \Pi_T = V_T = \int_0^T S_u du$$

$$= \cancel{\Pi_0} + \phi_T S_T - \cancel{\phi_0 S_0} - \int_0^T S_u d\phi_u$$

$$= \phi_T S_T - \int_0^T S_u d\phi_u \Rightarrow \begin{cases} \phi_T = 0 \\ d\phi_t = -dt \end{cases} \Rightarrow \phi_t = T - t$$

Question

Consider the Black-Scholes SDE, for μ, σ constants:

$$dS_t = \mu S_t dt + \sigma S_t dB_t.$$

(a). Solve the SDE.
(b). Find $\mathbb{E}\left[\frac{S_{t+\Delta t}}{S_t}\right]$ *and* $\mathbb{E}\left[(\frac{S_{t+\Delta t}}{S_t})^2\right]$

$d\langle S_t\rangle = \sigma^2 S_t^2 dt$

$= \text{"} dS_t \cdot dS_t \text{"}$ where $dB_t \cdot dB_t = dt$

(a). Consider $X_t = \ln S_t$

By Ito formula: $dX_t = \frac{1}{S_t} dS_t - \frac{1}{2}\frac{1}{S_t^2} d\langle S\rangle_t$

$= \frac{1}{S_t}(\mu S_t dt + \sigma S_t dB_t)$

$$-\frac{1}{2}\frac{1}{\cancel{S_t^2}}\sigma^2 \cancel{S_t^2}\, dt$$

$$\Rightarrow dX_t = \left(\mu - \frac{\sigma^2}{2}\right)dt + \sigma\, dB_t$$

$$\Rightarrow X_t = X_0 + \int_0^t \left(\mu - \frac{\sigma^2}{2}\right) ds + \int_0^t \sigma\, dB_s$$

$$= X_0 + \left(\mu - \frac{\sigma^2}{2}\right) t + \sigma B_t$$

$$\Rightarrow S_t = S_0 \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma B_t\right)$$

$$\Rightarrow \frac{S_t}{S_0} \sim LN\left(\left(\mu - \frac{\sigma^2}{2}\right)t,\ \sigma^2 t\right).$$

(b) $$\frac{S_{t+\Delta t}}{S_t} = \frac{S_0 \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)(t+\Delta t) + \sigma B_{t+\Delta t}\right)}{S_0 \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma B_t\right)}$$

$$= \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)\Delta t + \sigma\left(B_{t+\Delta t} - B_t\right)\right)$$

$$\Rightarrow \mathbb{E}\left[\frac{S_{t+\Delta t}}{S_t}\right] = \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)\Delta t\right) \cdot \underbrace{\mathbb{E}\left[e^{\sigma(B_{t+\Delta t} - B_t)}\right]}_{\mathbb{E}[e^{\sigma Z}]}$$

where $Z \sim N(0, \Delta t)$

$$= \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)\Delta t + \frac{\sigma^2}{2}\Delta t\right)$$

$$= \exp(\mu \Delta t)$$

$$\mathbb{E}\left[\left(\frac{S_{t+\Delta t}}{S_t}\right)^2\right] = \exp\left(2\cdot\left(\mu - \frac{\sigma^2}{2}\right)\Delta t\right) \cdot \underbrace{\mathbb{E}\left[e^{2\sigma(B_{t+\Delta t} - B_t)}\right]}_{\exp(2\sigma^2 \Delta t)}$$

$$= \exp(2\mu\Delta t + \sigma^2 \Delta t)$$

Question

Let $(B_t)_{t\geq 0}$ be a standard Brownian motion and $f : \mathbb{R} \to \mathbb{R}$ be given. Compute the value function $v : [0, T] \times \mathbb{R} \to \mathbb{R}$ defined by

$$v(t,x) = \mathbb{E}[f(B_T)|B_t = x] = \mathbb{E}[\, f(x + \underbrace{B_T - B_t}_{N(0,\, T-t)})\,]$$

when $f(x) = |x|$ (see solution of mid-term exam). For a general $f(x) \leq Ce^{|x|}$ for some $C > 0$, verify that v satisfies the heat equation

$$\partial_t v(t,x) + \frac{1}{2}\partial^2_{xx} v(t,x) = 0.$$

$$V(t,x) = \mathbb{E}[\,|x+z|\,] \quad z \sim N(0, T-t) \sim (T-t)\cdot N(0,1)$$

$$= \int_{\mathbb{R}} |x+z| \cdot \frac{1}{\sqrt{2\pi(T-t)}} \exp\left(-\frac{z^2}{2(T-t)}\right) dz$$

$$= \int_{-x}^{+\infty} \quad dz + \int_{-\infty}^{-x} \quad dz$$

$$\overset{V(t,x)}{\overset{\shortparallel}{\mathbb{E}}}\left[f(x+z) \right] \quad \text{where } z \sim N(0, T-t)$$

$$= \int_{\mathbb{R}} f(x+z) \cdot f_z(z)\, dz$$

$$= \int_{\mathbb{R}} f(x+z) \cdot \underbrace{\frac{1}{\sqrt{2\pi(T-t)}} \cdot \exp\left(-\frac{z^2}{2(T-t)}\right)} dz$$

$$\partial_t V(t,x) = \int_{\mathbb{R}} f(x+z)\, \partial_t \left(\downarrow \right) dz$$

$$= -\frac{1}{2} \mathbb{E}\left[\frac{z^2 + (T-t)}{(T-t)^2} \cdot f(x+z) \right]$$

$$= -\frac{1}{2} \partial^2_{xx} V(t,x)$$

Method 2, with Tutorial 5

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fix $h > 0$

$$V(t+h, B_{t+h}) = V(t, B_t) + \int_t^{t+h} \left(\partial_t V + \frac{1}{2} \partial^2_{xx} V \right)(s, B_s)\, ds$$

$$+ \int_t^{t+h} \partial_x V(s, B_s)\, dB_s$$

① Take expectation

② Send $h \to 0$

③ $\searrow$

$$\mathbb{E}\left[\left(\partial_t V + \frac{1}{2} \partial^2_{xx} V \right)(t, B_t) \,\middle|\, B_t = x \right] = 0$$

$$\Downarrow$$

$$\partial_t V + \frac{1}{2} \partial^2_{xx} V = 0$$

Method 3.

SDEs

Question (optional)

Solve the following SDEs, given μ, σ, a constants:

(a). (Ornstein-Uhlenbeck)

$$dX_t = \mu X_t dt + \sigma dB_t.$$

(b). (Vasicek)

CIR

$$dr_t = (a - r_t)dt + \sigma dB_t$$